AN ALGORITHM FOR THE SOLUTION OF LINEAR PROGRAMING PROBLEMS

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by

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ABSTRACT

Linear programming techniques are becoming of greater importance because the use of computerization has increased the fields for applications for linear programs. The primal-dual algorithm, in which the constraints are added one at a time, is investigated as a possible faster solution method. A computer program was developed to compare this method with the standard primal-dual algorithm using the full set of constraints at one time. Several random problems were solved using these two methods, and the results indicated a significant improvement in the solution time by the use of adding the constraints one at a time.

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I. INTRODUCTION

New linear programming algorithms have been developed to reduce the computational time in solving linear programs. The purpose of this thesis is to investigate the merits of one such new algorithm. This method consists of introducing the constraint equations one at a time. After each constraint is added, the "smaller", or submatrix, problem is solved using the primal-dual algorithm. This continues until all constraints have been added and a solution is obtained.

The rationale for this approach is that small matrices are used in the initial stages of solving the linear program; the size of the matrices increases only when additional constraints are introduced. If the number of iterations used in this method is not significantly different from the number of iterations used with the full matrices, the manipulation of the smaller matrices in the initial stages will reduce the solution time.

II. NOTATION

m number of constraint equations.

n number of legitimate variables.

A mxn matric of coefficients of the constraint equations with elements a_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n$.

P mxm matrix of the basis vectors.

P^{-1} inverse of the basis.

P_j mx1 column vector which is the j^{th} column of A.

$$\text{i.e., } P_j = \begin{bmatrix} a_{ij} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}, \quad j = 1,\ldots,n .$$

P_{ai} mx1 column vector associated with the i^{th} artificial variable, $i = 1,\ldots,m$.

x_j mx1 column vector with elements x_{ij}, where $X_j = P^{-1}P_j$.

C nx1 column vector with elements c_j which are the costs of the legitimate variables.

B mx1 column vector with elements b_i which are the right-hand sides of the constraint equations.

s_j dual slack variables.

$\hat{s}_j$ dual slack variables after a dual iteration.

$\bar{d}$ mx1 column vector whose elements are the coefficients of the basis variables of the added constraint equation.

Notation used with tableau:

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0-4	(1)	1/2	1/3	0	5/6	1	1	-1/6	$\theta = b_0 - 4$
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6	
z_j-c_j	$4-b_0$	[-1]	-1/2	-1/3	0	-5/6	-1	-------		$\hat{\theta} = -4/-1 = 4$
s_j	$4b_0$	4	2	4	0	4	4	-------		
$\hat{s}_j$	16	0	0	8/3	0	2/3	0	-------		
		↑								

↑ the vector to be introduced into the basis. e.g., in this tableau P_0 will be introduced.

(1) pivot element for a primal iteration, i.e., the θ criterion is $\theta = \min_i(x_{iB}/x_{ij})$ such that $x_{ij} > 0$.

[-1] pivot element for a dual iteration, i.e., the $\hat{\theta}$-criterion is $\hat{\theta} = \min_j(-s_j/z_j-c_j)$ such that $z_j-c_j < 0$.

III. FORMULATION OF THE PROBLEM AND SOLUTION PROCEDURE

The general linear programming problem is to maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \geq 0, \quad i = 1,\ldots,m , \tag{1}$$

and

$$x_j \geq 0, \quad j = 1,\ldots,n .$$

The <u>modified primal</u> uses an additional constraint

$$x_0 + \sum_{j=1}^{n} x_j = b_0,$$

where the cost of x_0 is zero and b_0 is arbitrarily large, so that for $x_0 > 0$ the constraint adds no additional restriction on (1).

The modified primal is written to maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} x_0 + \sum_{j=1}^{n} x_j &= b_0 , \\ \sum_{j=1}^{n} a_{ij} x_j &= b_i, \quad i = 1,\ldots,m , \end{aligned} \tag{2}$$

and

$$x_j \geq 0, \quad j = i,\ldots,n .$$

From (2) we can write the <u>modified dual</u> with slack variables, s_j, $j = 0,1,\ldots,n$, added. That is to minimize

$$w_0 b_0 + \sum_{i=1}^{m} w_i b_i$$

subject to

$$w_0 - s_0 = 0,$$

$$w_0 + \sum_{i=1}^{m} w_i a_{ij} - s_j = c_j, \quad j = 1,\ldots,n, \tag{3}$$

and

$$w_i \text{ unrestricted for } i = 0,1,\ldots,m.$$

The starting feasible solution to the primal dual algorithm is $w_i = 0$, $i = 1,\ldots,m$, and $w_0 = \max_j (c_j, 0)$.

For an optimal solution the complementary slackness condition must hold. That is

$$s_0 x_0 + \sum_{j=1}^{n} s_j x_j = 0 . \tag{4}$$

Adding artificial variables, x_{ai}, $i = 0,1,\ldots,m$, to (2) with the cost of the artificial variables set to -1 and the cost of the legitimate variables set to zero, the <u>extended primal</u> can be written as

maximize

$$- x_{a0} - \sum_{i=1}^{m} x_{ai}$$

subject to

$$x_0 + \sum_{j=1}^{n} x_j + x_{a0} = b_0 ,$$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{ai} = b_i, \quad i = 1,\ldots,m,$$

and

$$x_j \geq 0, \; j = 1,\ldots,n, \text{ and } x_{ai} \geq 0, \; i = 0,\ldots,m .$$

Solving the extended primal is similar to using a Phase I Revised Simplex method. [1] However, in the primal-dual algorithm, when Phase I ends the linear program is solved because complementary slackness is maintained throughout the solution procedure.

If a new constraint is added to the tableau, complementary slackness is maintained without changing the dual slack variables. This can be shown as follows:

> Assume we have a feasible solution to the problem with k constraint equations. This means that $s_j = 0$ for all j such that $P_j \in P$ and $x_j = 0$ for all j such that $P_j \notin P$. These conditions imply that complementary slackness is maintained, and that the modified dual also has a feasible solution.
>
> Now we add the $k + 1^{st}$ constraint which introduces a new dual variable, w_{k+1}, but no new dual slack variables, s_j, $j = 0,1,\ldots,n$. We need a feasible solution to the modified dual for the enlarged system. Observe that we have a feasible solution if we set $w_{k+1} = 0$ since then the s_j, $j = 0,1,\ldots,n$, remain unchanged. In particular, $s_j = 0$ for all j such that $P_j \in P$, that is, for the legitimate variables. Also, $x_j = 0$ for all j such that $P_j \notin P$, which implies that we have maintained complementary slackness.

It is worth noting that the $z_j - c_j$, $j = 0,1,\ldots,n$, must be recalculated since the new constraint which is added to the

extended primal starts with its artificial variable, $x_{a,k+1}$, in the basis with its cost set at -1.

The new basis is $\bar{P} = \begin{matrix} P & \bar{0} \\ \bar{d}^T & 1 \end{matrix}$, where $P_{a,k+1} = \begin{matrix} \bar{0} \\ 1 \end{matrix}$ is the artificial vector associated with $x_{a,k+1}$. Now we can solve the new k + 1 system using the primal-dual algorithm since complementary slackness has been maintained.

An optimal solution exists if and only if the following criteria are satisfied:

1. $z_j - c_j \geq 0$ for $j = 0,1,\ldots,n$,
2. $z_B - c_B = 0$, and
3. $x_0 > 0$.

The solution procedure is as follows:

The first tableau is set up using the first two constraints of the extended primal and the starting solution to the modified dual, which implies that at least one $s_j = 0$.

Step 1. Is there a j, say j_0, such that $s_{j0} = 0$ and $z_{j0} - c_{j0} < 0$?

a. Yes. Go to 2.

b. No. Go to 3.

Step 2. Introduce P_{j0} into the basis using the minimum θ-criterion and a primal iteration. Since the extended primal is bounded a pivot will always exist. Note that the s_j remain unchanged for all j. Go to 1.

Step 3. Is $z_j - c_j < 0$ for some j?

a. Yes. Go to 4.

b. No. Go to 5.

Step 4. Use the minimum $\hat{\theta}$-criterion. Is $\hat{\theta}$ bounded?

a. Yes. Perform a dual iteration to compute a new set of s_j's, say $\hat{s}_j$'s. Go to 1.

b. No. The linear program has no feasible solution. Stop.

Step 5. Is $z_B - c_B < 0$?

a. Yes. The linear program has no feasible solution. Stop.

b. No. Go to 6.

Step 6. Have all of the constraints been added?

a. Yes. Go to 9.

b. No. Go to 7.

Step 7. Introduce the next restraint, say the $k + 1^{st}$. Place the artificial vector $P_{a,k+1}$ in the basis. Compute $x_{B,k+1}$. Is $x_{B,k+1} \geq 0$?

a. Yes. Go to 8.

b. No. Multiply all coefficients of the new constraint, except for the artificial variable, by -1. This assures that $x_{B,k+1} \geq 0$ and the artificial variable is non-negative. Go to 8.

Step 8. For the system with k + 1 restraints, compute the new values of $z_j - c_j$ for $j = 0,1,\ldots,n$. Go to 1.

Step 9. Is $x_0 = 0$?

a. Yes. The linear program is unbounded. Stop.

b. No. An optimal solution has been found. Stop.

IV. SAMPLE PROBLEM

Consider the following example:

maximize

$$z = 2x_1 + 4x_3$$

subject to

$$3x_1 + 4x_2 + 6x_3 + x_4 = 24 ,$$

$$4x_1 + 3x_2 + 12x_3 + x_5 = 24 ,$$

$$x_1 + x_2 + 4x_3 = 8 ,$$

and

$$x_j \geq 0, \; j = 1,\ldots,5.$$

Then the extended primal is

maximize

$$- x_{a0} - x_{a1} - x_{a2} - x_{a3}$$

subject to

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_{a0} = b_0,$$

$$3x_1 + 4x_2 + 6x_3 + x_4 + x_{a1} = 24,$$

$$4x_1 + 3x_2 + 12x_3 + x_5 + x_{a2} = 24,$$

$$x_1 + x_2 + 4x_3 + x_{a3} = 8,$$

$x_j \geq 0$ for $j = 0,\ldots,5$, and $x_{ai} \geq 0$ for $i = 0,\ldots,3$.

The dual slack variables are $s_0 = \max_j(c_j,0) = 4$ with $j = 3$.

Then $s_B = x_0 b_0 = 4b_0$, and $s_j = s_0 - c_j$ for $j = 1,\ldots,5$, so that $s_1 = 2$, $s_2 = 4$, $s_3 = 0$, $s_4 = 4$, and $s_5 = 4$.

The starting tableau, using the first original constraint, is

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0	1	1	1	1	1	1	1	0	
P_{a1}	24	0	3	4	(6)	1	0	0	1	$\theta = 24/6 = 4$
z_j-c_j	$-b_0-24$	-1	-4	-5	[-7]	-2	-1	---	---	
s_j	$4b_0$	4	2	4	0	4	4	---	---	

↑

From step 1, we see that $s_3 = 0$ and $z_3-c_3 < 0$. Using the minimum θ-criterion (as discussed in section III) in step 2, we introduce P_3 into the basis and remove P_{a1} from the basis.

Since there is no j_0 for which $s_{j0} = 0$ and $z_{j0}-c_{j0} < 0$, but $z_j-c_j < 0$ for several values of j, we arrive at step 4. Using the minimum $\hat{\theta}$-criterion a new set of s_j's, called $\hat{s}_j$'s are calculated.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	
P_{a0}	b_0-4	(1)	1/2	1/3	0	5/6	1	1	-1/6	$\theta = b_0-4$
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6	
z_j-c_j	$-b_0+4$	[-1]	-1/2	-1/3	0	-5/6	-1	---	---	$\hat{\theta}= -4/-1 = 4$
s_j	$4b_0$	4	2	4	0	4	4	---	---	
$\hat{s}_j$	16	0	0	8/3	0	2/3	0	---	---	

↑

Now $\hat{s}_0 = 0$ and $z_0-c_0 < 0$ so, from step 2, we introduce P_0 into the basis and remove P_{a0} from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}
P_0	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6
P_3	4	0	1/2	2/3	1	1/6	0	0	1/6
z_j-c_j	0	0	0	0	0	0	0	---	---
s_j	16	0	0	8/3	0	2/3	0	---	---

From the above tableau we trace through steps 1b, 3b, 5b, 6b, and arrive at step 7. Note that we have obtained an optimal solution to the subproblem with one constraint. In step 7 we introduce the second constraint. Since P_0 and P_3 are basis vectors, $\bar{d}^T = (a_{20}, a_{23}) = (0,12)$; the new basis consists of P_0, P_3 and P_{a2}. With this basis we find that $x_{B2} = -24 < 0$ so that step 7b must be used. The second restraint is replaced by

$$-4x_{1} - 3x_{2} - 12x_{3} - x_5 + x_{a2} = -24 ,$$

which is used throughout the remainder of the solution procedure. Note that now $\bar{d}^T = (0,-12)$ and $x_{B2} = 24 > 0$. New values of x_j-c_j for $j = 0,1,\ldots,n$ are computed (step 8), and the new tableau is

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	
P_0	b_0-4	1	1/2	1/3	0	5/6	1	1	-1/6	0	
P_3	4	0	(1/2)	2/3	1	1/6	0	0	1/6	0	$\theta = \frac{4}{1/2} = 8$
P_{a2}	24	0	2	5	0	2	-1	0	2	1	
z_j-c_j	-24	0	[-2]	-5	0	-2	1	---	---	---	
s_j	16	0	0	8/3	0	2/3	0	---	---	---	

↑

Since $s_1 = 0$ and $z_1-c_1 < 0$ (step 1) we go to step 2. Using a primal iteration, we introduce P_1 into the basis and eliminate P_3 from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	
P_0	b_0-8	1	0	-1/3	-1	2/3	1	1	-1/3	0	
P_1	8	0	1	4/3	2	1/3	0	0	1/3	0	
P_{a2}	8	0	0	7/3	-4	(4/3)	-1	0	4/3	1	$\theta = \frac{8}{4/3} = 6$
z_j-c_j	-8	0	0	-7/3	4	[-4/3]	1	---	---	---	$\hat{\theta} = \frac{-2/3}{-4/3} = 1/2$
s_j	16	0	0	8/3	0	2/3	0	---	---	---	
$\hat{s}_j$	12	0	0	3/2	2	0	1/2	---	---	---	

↑

Now $z_j-c_j \geq 0$ for all j for which $s_j = 0$. From steps 1b, 3a, and 4a, a new set of s_j's are computed. Then $\hat{s}_4$ becomes zero and $z_4-c_4 < 0$ so, from step 2, P_4 enters the basis and P_{a2} is removed from the basis.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}
P_0	b_0-12	1	0	-3/2	1	0	3/2	1	-1	-1/2
P_1	6	0	1	3/4	3	0	1/4	0	0	-1/4
P_4	6	0	0	7/4	-3	1	-3/4	0	1	3/4
z_j-c_j	0	0	0	0	0	0	0	---	---	---
s_j	12	0	0	3/2	2	0	1/2	---	---	---

Optimality has now been obtained with the second restraint added. Following steps 1b, 3b, 5b, 6b, and 7, we introduce

the third and final restraint. Since the basis vectors were P_0, P_1, and P_4, $\bar{d}^T = (a_{30}, a_{31}, a_{34}) = (0,1,0)$. The new basis vectors are P_0, P_1, P_4, P_{a3}. We find that $x_{B3} = 2 > 0$ so we go to step 8 and recompute $z_j - c_j$ for $j = 0,1,\ldots,n$.

The next sequence of steps is 1b, 3a, and 4a, which leads to a new set of s_j's.

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	P_{a3}	
P_0	b_0-12	1	0	-3/2	1	0	3/2	1	-1	-1/2	0	
P_1	6	0	1	3/4	3	0	1/4	0	0	-1/4	0	
P_4	6	0	0	7/4	-3	1	-3/4	0	1	3/4	0	
P_{a3}	2	0	0	1/4	(1)	0	-1/4	0	0	1/4	1	$\theta = 2$
$z_j - c_j$	-2	0	0	-1/4	[-1]	0	1/4	---	---	---	---	$\theta = -2/-1 = 2$
s_j	12	0	0	3/2	2	0	1/2	---	---	---	---	
$\hat{s}_j$	8	0	0	1	0	0	1	---	---	---	---	

↑

Steps 1a and 2 bring P_3 into the basis with the elimination of P_{a3} from the basis. The new tableau is:

P	B	P_0	P_1	P_2	P_3	P_4	P_5	P_{a0}	P_{a1}	P_{a2}	P_{a3}
P_0	b_0-14	1	0	-7/4	0	0	7/4	1	-1	-3/4	-1
P_1	0	0	1	0	0	0	1	0	0	- 1	-3
P_4	12	0	0	5/2	0	1	-3/2	0	1	3/2	3
P_3	2	0	0	1/4	1	0	-1/4	0	0	1/4	1
$z_j - c_j$	0	0	0	0	0	0	0	---	---	---	---
s_j	8	0	0	1	0	0	1	---	---	---	---

This tableau is the final tableau since the sequence of steps 1b, 3b, 5b, 6a, and 9b inform us that we have found an optimal solution to the original linear program. Note that the complementary slackness condition has been maintained, that is, $s_0 x_0 + \sum_{j=1}^{n} s_j x_j = 0$. Therefore, the optimal solution is

$$x_0 = b_0 - 14 > 0,$$

$$x_1 = x_2 = x_5 = 0,$$

$$x_3 = 2, \text{ and}$$

$$x_4 = 12,$$

with the optimal cost $z = s_B = 8$.

V. PROGRAMMING TECHNIQUE

The linear programming technique described in this thesis was programmed in FORTRAN IV for use on the IBM 360/67 computer. One subroutine is used for the primal simplex iteration and another is used for the dual iteration. The final subroutine is used for the addition of constraints. The main (driving) routine is used to solve both the full linear program and the linear program using addition of constraints as described in this thesis. Using the main routine to solve both problems eliminates any time differences due to differences in programming techniques. Both of the solution procedures were timed*, and the number of iterations of each were counted. Read and print times were not included in the timing.

*The timing routine was developed by Lt. E.A. Singer, a student at the Naval Postgraduate School.

VI. EFFICIENCY OF THE ALGORITHM

To eliminate the considerable time and effort required to input data by hand, a subroutine was designed which generates random problems of a large size. This routine uses a random number generator to produce the elements of the A, B, and C matrices. The following criteria were used in order to insure the existence of a bounded feasible solution:

maximize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - x_{si} = b_i, \quad i = 1,\ldots,m,$$

and

$$c_j \leq 0, \; x_j \geq 0, \; x_{si} \geq 0, \; a_{ij} \geq 0, \; b_i \geq 0$$

$$\text{for } i = 1,\ldots,m, \; j = 1,\ldots,n,$$

where x_{si} is the slack variable for the i^{th} constraint.

For this investigation the subroutine generated problems having 70 variables (including 20 slack variables), 20 constraint equations, and 50 cost coefficients using the following uniform distributions:

a_{ij}, uniform (0,1);

b_i, uniform (0,5);

c_j, uniform (-1,0).

A total of 40 problems were solved using the random problem generator described above. The execution times for these problems are given in Table I at the end of this section.

The comparison of solution times shows that all 40 problems ran faster using the method described in this thesis, than with the standard primal-dual algorithm. The time differences range from 15.17 seconds to 48.18 seconds with an average time difference of 27.53 seconds.

TABLE I

Prob. No.	Full Array		Addition of Constraints		
	Time (x_i) (sec.)	Iter.	Time (y_i) (sec.)	Iter	$x_i - y_i$
1	87.82	23	56.50	21	31.32
2	87.84	23	56.91	23	30.93
3	80.22	21	56.53	21	23.69
4	87.93	23	56.70	22	31.23
5	84.06	22	58.61	23	25.45
6	91.63	24	56.62	22	35.01
7	95.45	25	58.65	23	36.80
8	80.31	21	56.52	21	23.79
9	99.60	26	65.67	32	33.93
10	80.16	21	56.51	21	23.65
11	80.18	21	56.50	21	23.68
12	91.65	24	65.12	29	26.53
13	80.19	21	57.99	22	22.20
14	91.74	24	64.35	30	27.39
15	80.20	21	56.48	21	23.72
16	84.02	22	57.10	22	26.92
17	95.52	25	59.36	24	36.16
18	83.95	22	58.71	22	25.24
19	84.14	22	65.47	25	18.67
20	99.25	26	70.03	28	29.22
21	107.08	28	58.90	25	48.18
22	87.85	23	56.51	21	31.34
23	84.01	22	61.39	24	22.62
24	80.16	21	56.52	21	23.64
25	80.17	21	56.49	21	23.68
26	80.16	21	56.50	21	23.66
27	80.19	21	56.52	21	23.67
28	84.21	22	58.84	23	25.37
29	103.27	27	57.85	24	45.42
30	87.83	23	59.47	23	28.36
31	84.07	22	59.33	22	24.74
32	88.19	23	56.92	23	31.27
33	80.28	21	56.56	21	23.72
34	87.96	23	56.56	21	31.40
35	84.11	22	57.09	23	27.02
36	84.05	22	56.56	21	27.49
37	80.20	21	56.51	21	23.69
38	80.18	21	56.54	21	23.64
39	84.03	22	68.86	26	15.17
40	80.29	21	56.54	21	23.75
Total	3454.01		2352.77		1101.24
Average	86.35		58.82		27.53

VII. SUMMARY AND CONCLUSIONS

A modification of the primal-dual algorithm has been presented. This modification differs from the standard primal-dual algorithm in that the constraint equations are introduced one at a time, and each subproblem is solved before the next constraint is added.

This algorithm was programmed in FORTRAN IV for the IBM 360/67. The program was designed so that any given linear program is solved first by the standard primal-dual algorithm, and then is resolved using the modified primal-dual procedure. Further, the same subroutines are used for both methods in order to eliminate timing bias due to coding differences. In fact the modified procedure contains steps which are not included in the timing of the standard routine.

In all cases the modified procedure was faster than the standard procedure. Of 40 test problems the standard method averaged 86.35 seconds per problem, whereas the modified method averaged 58.82 seconds per problem. One should not judge the actual running time of the test problems since no attempt was made to improve the efficiency of the computer program on an absolute basis; only the relative speeds of the two methods is of importance.

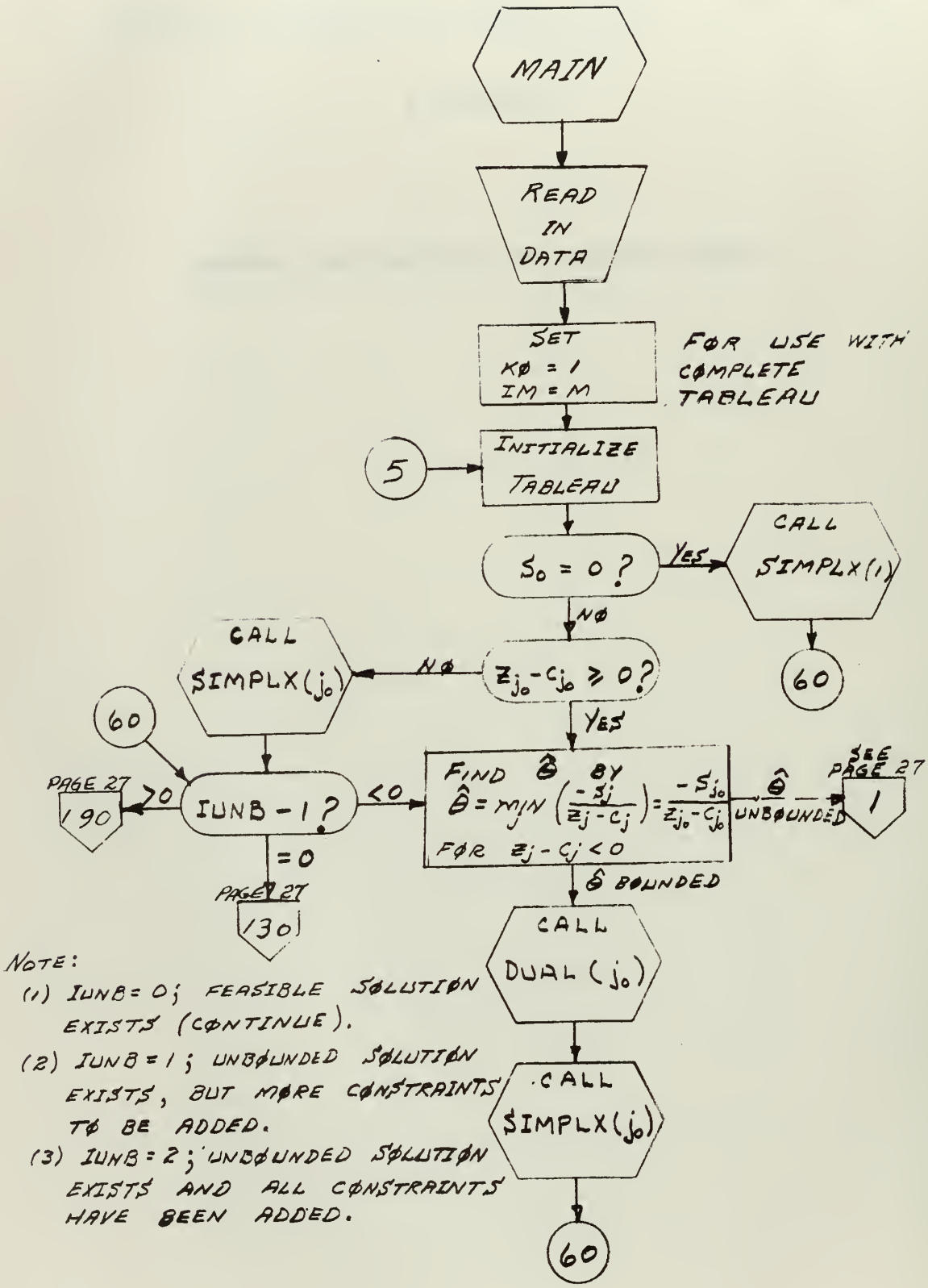
BIBLIOGRAPHY

1. Hadley, G., Linear Programming. Reading, Mass: Addison-Wesley Publishing Co., Inc., 1962.

APPENDIX A

FLOW DIAGRAMS OF THE COMPUTER PROGRAM

1. MAIN PROGRAM

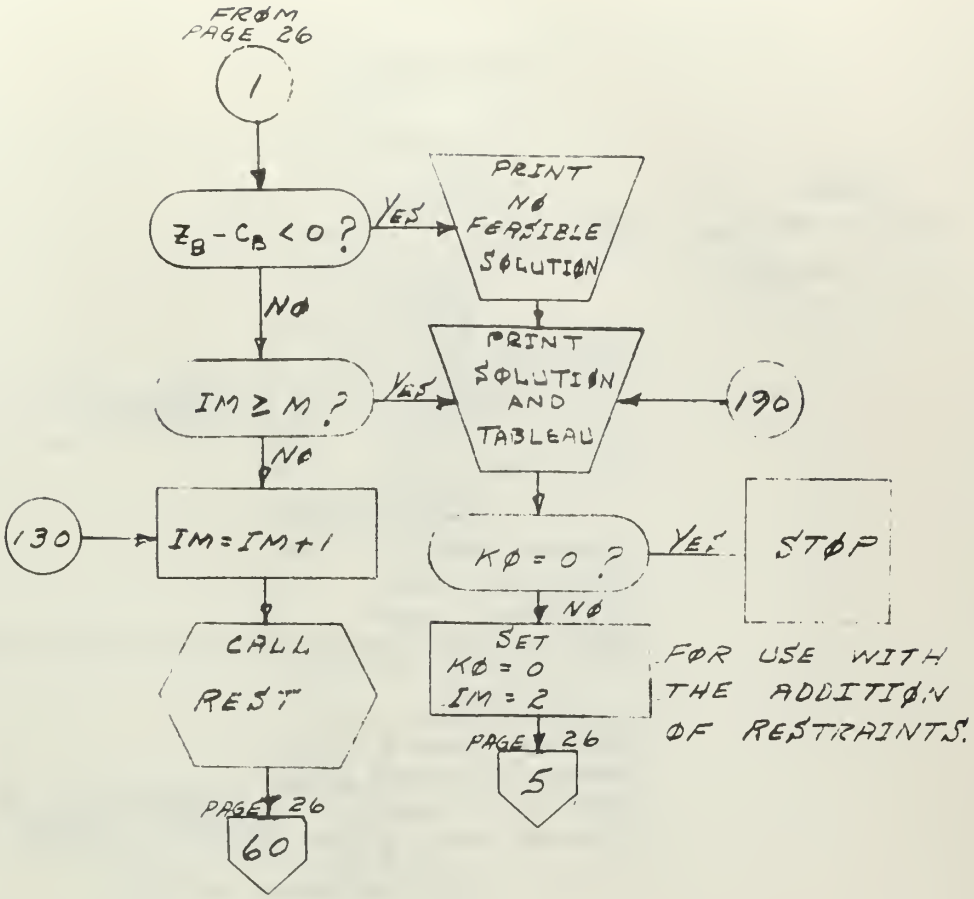


NOTE:

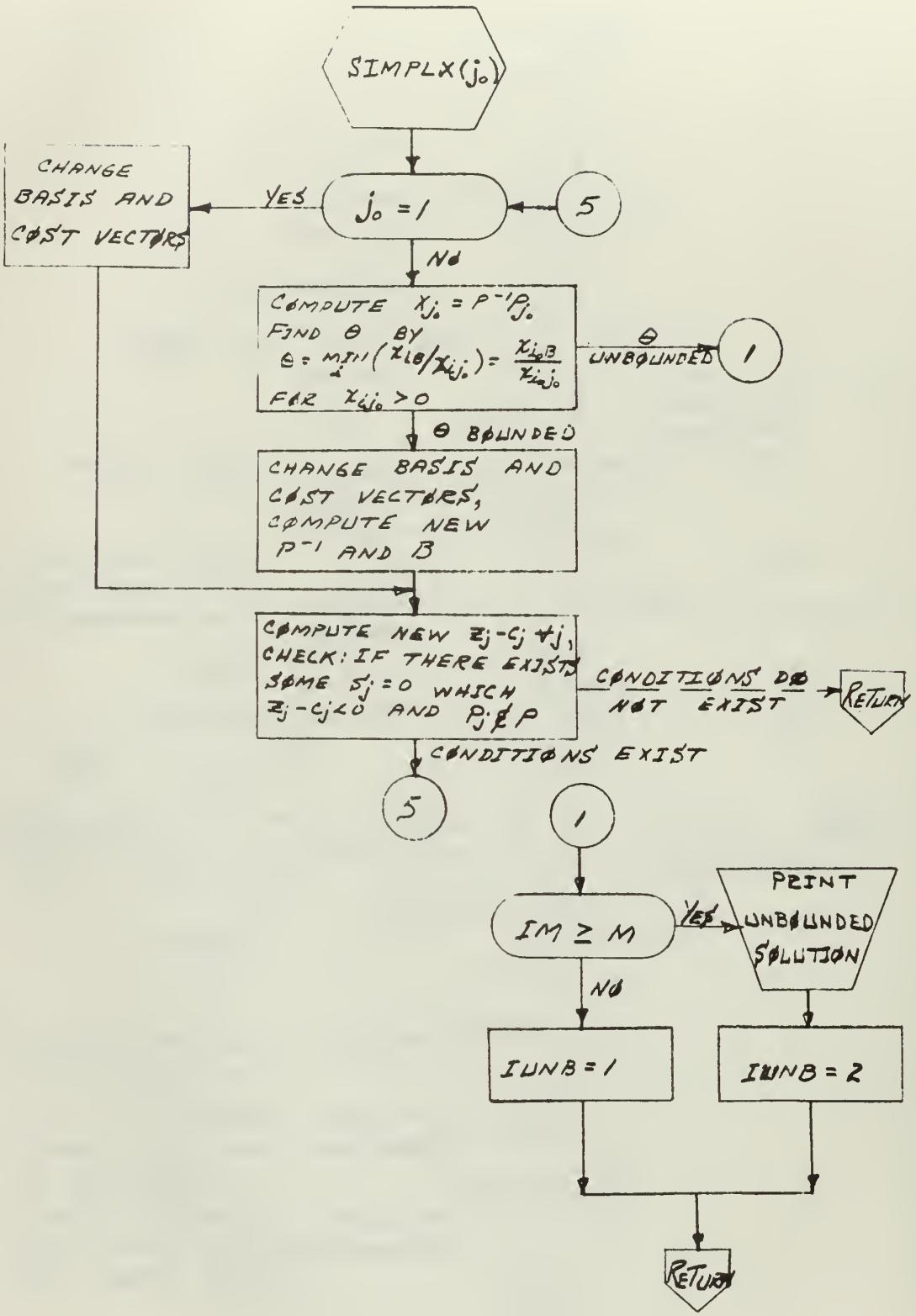
(1) IUNB = 0; FEASIBLE SØLUTIØN EXISTS (CØNTINUE).

(2) IUNB = 1; UNBØUNDED SØLUTIØN EXISTS, BUT MØRE CØNSTRAINTS TØ BE ADDED.

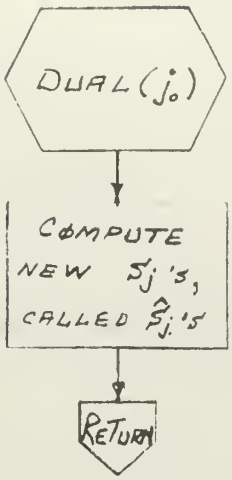
(3) IUNB = 2; UNBØUNDED SØLUTIØN EXISTS AND ALL CØNSTRAINTS HAVE BEEN ADDED.


FROM PAGE 26
1
$Z_B - C_B < 0$?
YES
PRINT NO FEASIBLE SOLUTION
NO
IM ≥ M ?
YES
PRINT SOLUTION AND TABLEAU
190
NO
130
IM = IM + 1
KØ = 0 ?
YES
STØP
NO
CALL REST
SET KØ = 0 IM = 2
FOR USE WITH THE ADDITION OF RESTRAINTS.
PAGE 26
60
PAGE 26
5

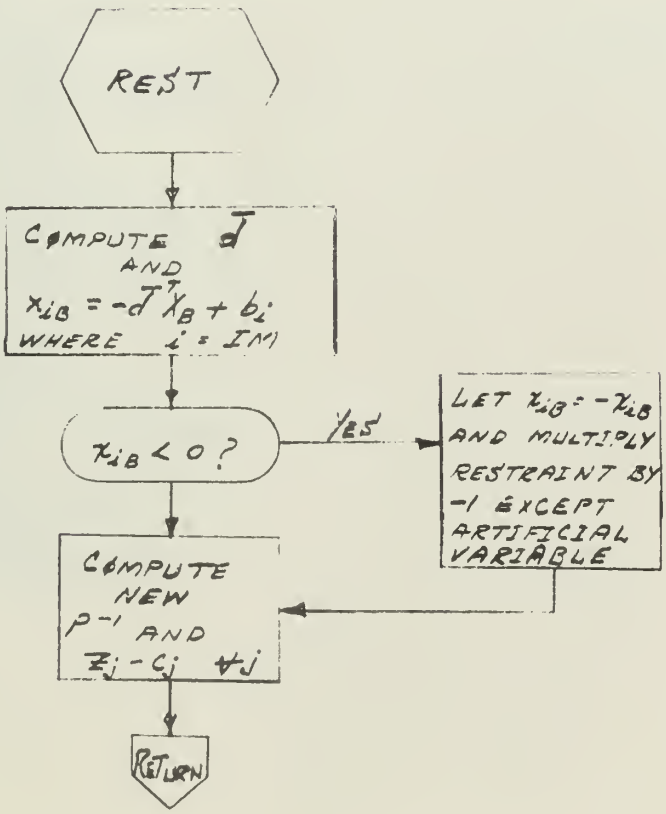
2. SIMPLEX ITERATION SUBROUTINE



3. DUAL ITERATION SUBROUTINE



4. ADDITION OF NEW RESTRAINT SUBROUTINE



APPENDIX B

FORTRAN LISTING OF THE COMPUTER PROGRAM

```
0001            DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),  0000020
                1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)      0000030
0002             COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                   0000040
0003             COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP       0000045
                C
                C NOTE - WS(J) IN THIS PROGRAM AND S(J) IN THESIS ARE EQUIVALENT
                C
0004             EP = 10.0E-6                                                       0000050
0005             XE = URN(0)                                                        0000055
0006             IPROB = 0                                                          0000060
0007           6 IF(IPROB.GE.50) GO TO 500                                          0000070
0008             IPROB = IPROB +1                                                   0000080
0009             CALL PSEUDO                                                        0000080
0010             M = M + 1                                                          0000090
0011             N = N + 1                                                          0000100
                C
                C KO=1 IS FOR USE OF COMPLETE TABLEAU
                C
0012             KO = 1                                                             0000260
0013             IM = M                                                             0000265
                C
                C INITIALIZE TABLEAU
                C
0014           5 DO 1 I=2,M                                                         0000270
0015             BT(I,1) = 0.0                                                      0000280
0016             BT(I,2) = B(I-1)                                                   0000290
0017           1 P(I,1) = 0.0                                                       0000300
0018             ITER = 1                                                           0000305
0019             DO 2 J=1,N                                                         0000310
0020           2 P(1,J) = 1.0                                                       0000320
0021             BT(1,1) = 1.0                                                      0000325
0022             BT(1,2) = 0.0                                                      0000330
0023             DO 3 I=1,M                                                         0000340
0024             IBAS(I) = N+I                                                      0000350
0025           3 ONE(I) = -1.0                                                      0000360
0026             DO 10 I=1,M                                                        0000370
0027             DO 10 J=1,M                                                        0000380
0028             IF(I.EQ.J) GO TO 9                                                 0000390
0029             PINV(I,J) = 0.0                                                    0000400
0030             GO TO 10                                                           0000410
0031           9 PINV(I,J) = 1.0                                                    0000420
0032          10 CONTINUE                                                         0000430
0033             CALL TIMEIT(C,CL)                                                  0000435
0034             ZCB(1) = -1.0                                                      0000440
0035             ZCB(2) = 0.0                                                       0000450
0036             DO 11 I=2,IM                                                       0000460
0037          11 ZCB(2) = ZCB(2) -BT(I,2)                                           0000470
0038             ZC(1) = -1.0                                                       0000480
0039             DO 20 J =2,N                                                       0000490
0040             ZC(J) = -1.C                                                       0000500
0041             DO 20 I = 2,IM                                                     0000510
0042          20 ZC(J) = ZC(J) -P(I,J)                                              0000520
0043             IND = 0                                                            0000530
0044             WS(1) = 0.0                                                        0000540
0045             DO 30 J=2,N                                                        0000550
0046             IF(C(J).LE.WS(1))GO TO 30                                          0000560
0047             WS(1) = C(J)                                                       0000570
0048             IND = J                                                            0000580
0049          30 CONTINUE                                                         0000590
0050             WSB(1) = WS(1)                                                     0000600
0051             WSB(2) = 0.0                                                       0000610
0052             DO 40 J=2,N                                                        0000620
0053          40 WS(J) = WS(1) -C(J)                                                0000630
                C
                C DETERMINE VECTOR TO INTRODUCE
                C
0054             IF(IND.EQ.0.0)GO TO 50                                             0000640
0055             IF(ZC(IND).GE.0.0)GO TO 70                                         0000650
0056             CALL SIMPLX(IND)                                                   0000660
0057             GO TO 60                                                           0000670
0058          50 CALL SIMPLX(1)                                                     0000680
0059          60 IF(IUNB-1)70,130,190                                               0000690
0060          70 TEMP = 999999.                                                     0000700
```

```
C
C FIND THETA FOR DUAL = MIN(-S(J)/ZC(J))OVER J FOR ZC(J)<0
C
      IND = 0                                                          0000710
      DO 80 J=1,N                                                      0000720
      IF(ZC(J).GE.0.0)GO TO 80                                         0000730
      DO 75 I=1,IM
      IF(IBAS(I).EQ.J) GO TO 80
   75 CONTINUE
      XE= -WS(J)/ZC(J)                                                 0000740
      IF(TEMP.LE.XE)GO TO 80                                           0000750
      TEMP = XE                                                        0000760
      IND = J                                                          0000770
   80 CONTINUE                                                         0000780
      IF(IND.EQ.0) GO TO 100                                           0000790
      CALL DUAL(IND)                                                   0000800
      CALL SIMPLX(IND)                                                 0000810
      GO TO 60                                                         0000820
C
C CHECK FOR INFEASIABILITY
C
  100 IF(ZCB(1))200,110,120                                            0000830
  110 IF(ZCB(2).LT.0.0) GO TO 200                                      0000840
  120 IF(IM.GE.M) GO TO 190                                            0000850
C
C ADD NEXT RESTRAINT
C
  130 IM = IM + 1                                                      0000860
      CALL REST                                                        0000870
      IUNB = 0                                                         0000880
      GO TO 60                                                         0000890
  190 CALL TIMEIT(-1,CL)                                               0000900
      WRITE(6,5000)IPROB,CL,ITER                                       0000910
 5000FORMAT(10X,'PROBLEM',I4,'  TIME IS ',-6PF15.6,'  SECONDS WITH',   0000920
     1I6,'  ITERATIONS',//)                                            0000940
      IF(KO.EQ.0)GO TO 6                                               0000990
C
C KO=0 IS FOR USE OF ADDITION OF RESTRAINTS
C
      IM=2                                                             0001000
      KO=0                                                             0001010
      GO TO 5                                                          0001020
  500 STOP                                                             0001030
  200 WRITE(6,4010)                                                    0001040
      GO TO 190                                                        0001050
 4010 FORMAT(10X,'SOLUTION INFEASIBLE'//)                              0001070
      END                                                              0001080
```

```
      C
      C
      C
0001        SUBROUTINE SIMPLX(J)                                                  0001090
0002       ODIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),    0001100
           1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)         0001110
0003        COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                      0001120
0004        COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP          0001125
0005        IUNB = 0                                                              0001130
      C
      C FOR J=1 THE ONLY CHANGE IS THE BASIS AND ZC(J)'S
      C
0006     5 IF(J.EQ.1) GO TO 130                                                   0001140
      C
      C COMPUTE X(J) AND FIND THETA FOR PRIMAL = MIN(X(I,B)/X(I,J)) OVER J
      C    FOR X(I,J)>0
      C
0007        T(1) = 999999.                                                        0001150
0008        T(2) = T(1)                                                           0001160
0009        DO 20 I=1,IM                                                          0001170
0010        L = IM-I+1                                                            0001180
0011        X(L) = 0.0                                                            0001190
0012        DO 10 K=1,IM                                                          0001200
0013        Y = PINV(L,K)*P(K,J)                                                  0001205
0014        X(L) = X(L) + Y                                                       0001210
0015     10 CALL ROUND(X(L),Y,EP)                                                 0001211
0016        IF(X(L).LE.0.)GO TO 20                                                0001215
0017        TEM(1) = BT(L,1)/X(L)                                                 0001220
0018        TEM(2) = BT(L,2)/X(L)                                                 0001230
0019        IF(T(1)-TEM(1))20,11,12                                               0001240
0020     11 IF(T(2).LE.TEM(2))GO TO 20                                            0001250
0021     12 T(1) = TEM(1)                                                         0001260
0022        T(2) = TEM(2)                                                         0001270
0023        ID = L                                                                0001280
0024     20 CONTINUE                                                              0001290
      C
      C COMPUTE NEW TABLEAU
      C
0025        IF(T(1).EQ.999999.)GO TO 50                                           0001300
0026        IBAS(ID) = J                                                          0001310
0027        ONE(ID) = 0.0                                                         0001320
0028        DO 30 I=1,IM                                                          0001330
0029        IF(I.EQ.ID) GO TO 30                                                  0001333
0030        Y = T(1)*X(I)                                                         0001333
0031        BT(I,1) = BT(I,1) - Y                                                 0001334
0032        CALL ROUND(BT(I,1),Y,EP)                                              0001335
0033        Y = T(2)*X(I)                                                         0001336
0034        BT(I,2) = BT(I,2) - Y                                                 0001336
0035        CALL ROUND(BT(I,2),Y,EP)                                              0001337
0036        XE = X(I)/X(ID)                                                       0001360
0037        DO 30 K=1,IM                                                          0001370
0038        Y = PINV(ID,K)*XE                                                     0001371
0039        PINV(I,K) = PINV(I,K) - Y                                             0001372
0040        CALL ROUND(PINV(I,K),Y,EP)                                            0001375
0041     30 CONTINUE                                                              0001385
0042        BT(ID,1) = BT(ID,1)/X(ID)                                             0001387
0043        BT(ID,2) = BT(ID,2)/X(ID)                                             0001388
0044        DO 31 I=1,IM                                                          0001386
0045     31 PINV(ID,I) = PINV(ID,I)/X(ID)                                         0001389
0046     70 ZCB(1) = 0.0                                                          0001390
0047        ZCB(2) = 0.0                                                          0001410
0048        DO 80 I=1,IM                                                          0001420
0049        ZCB(1) =ZCB(1) +ONE(I)*BT(I,1)                                        0001430
0050     80 ZCB(2) =ZCB(2) +ONE(I)*BT(I,2)                                        0001440
0051        ZC(1) = ONE(1)                                                        0001450
0052        DO 90 L=2,N                                                           0001460
0053        ZC(L) = 0.                                                            0001470
0054        DO 90 I=1,IM                                                          0001490
0055        DO 90 K=1,IM                                                          0001500
0056        Y = ONE(I)*PINV(I,K)*P(K,L)                                           0001505
0057        ZC(L) = ZC(L) + Y                                                     0001510
0058     90 CALL ROUND(ZC(L),Y,EP)                                                0001515
      C
```

```
C CHECK FOR AN ADDITION ITERATION WITH PRIMAL
C
0059        TEMP = 0.0                                                    0001520
0060        DO 120 L=1,N                                                  0001530
0061        IF(WS(L).NE.0.0) GO TO 120                                    0001540
0062        DO 110 I=1,IM                                                 0001550
0063        IF(IBAS(I).EQ.L)GO TO 120                                     0001560
0064      110 CONTINUE                                                    0001570
0065        IF(ZC(L).GE.TEMP) GO TO 120                                   0001580
0066        J=L                                                           0001590
0067        TEMP = ZC(L)                                                  0001600
0068      120 CONTINUE                                                    0001610
0069        ITER = ITER + 1                                               0001615
0070        IF(TEMP.NE.0.0) GO TO 5                                       0001620
0071      200 RETURN                                                      0001630
0072      130 IBAS(1) = 1                                                 0001640
0073        ONE(1) = 0.0                                                  0001650
0074        ID = 1                                                        0001655
0075        GO TO 70                                                      0001660
C
C SOLUTION UNBOUNDED
C
0076       50 IF(IM.GE.M) GO TO 60                                        0001670
0077          IUNB = 1                                                   0001680
0078          GO TO 200                                                  0001690
0079       60 WRITE(6,4020)                                              0001700
0080          IUNB = 2                                                   0001710
0081     4020 FORMAT(10X,'SOLUTION UNBOUNDED'//)                         0001720
0082          GO TO 200                                                  0001730
0083          END                                                        0001740
```

```
C
C
C
      SUBROUTINE DUAL(J)                                                0001750
      DIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2),0001760
     1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)     0001770
      COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                  0001780
      COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP      0001785
C
C COMPUTE NEW WS(J) WHICH IS EQUIVALENT TO S(J)
C
      XE = WS(J)/ZC(J)                                                  0001790
      DO 10 K=1,N                                                       0001800
      Y = ZC(K)*XE                                                      0001801
      WS(K) = WS(K) - Y                                                 0001802
   10 CALL ROUND(WS(K),Y,EP)                                            0001803
      Y = ZCB(1)*XE                                                     0001804
      WSB(1) = WSB(1) - Y                                               0001805
      CALL ROUND(WSB(1),Y,EP)                                           0001806
      Y = ZCB(2)*XE                                                     0001807
      WSB(2) = WSB(2) - Y                                               0001808
      CALL ROUND(WSB(2),Y,EP)                                           0001810
      RETURN                                                            0001840
      END                                                               0001850
```

```
C
C
C
      SUBROUTINE REST                                                   0001860
     ODIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2), 0001870
     1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)     0001880
      COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                  0001890
      COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP      0001895
C
C COMPUTE D-BAR AND B VECTOR
C
      K = IM-1                                                          0001900
      BT(IM,1) = 0.0                                                    0001910
      BT(IM,2) = B(K)                                                   0001920
      DO 10 I=1,K                                                       0001930
      IF(IBAS(I).GT.N) GO TO 5                                          0001935
      L = IBAS(I)                                                       0001940
      DBAR(I) = -P(IM,L)                                                0001950
      BT(IM,1) = BT(IM,1) + DBAR(I)*BT(I,1)                             0001960
      BT(IM,2) = BT(IM,2) + DBAR(I)*BT(I,2)                             0001970
      GO TO 10                                                          0001971
    5 DBAR(I) = 0.0                                                     0001972
   10 CONTINUE                                                          0001973
      IF(BT(IM,1))50,20,30                                              0001980
   20 IF(BT(IM,2).LT.0.0) GO TO 50                                      0001990
C
C COMPUTE NEW INVERSE AND COST VECTOR
C
   30 DO 40 J=1,K                                                       0002000
      DO 40 I=1,K                                                       0002010
   40 PINV(IM,J) = PINV(IM,J) + DBAR(I)*PINV(I,J)                       0002020
      ZC(1) = ONE(1)                                                    0002021
      DO 45 J=2,N                                                       0002022
      ZC(J) =0.0                                                        0002023
      DO 45 I=1,IM                                                      0002024
      DO 45 IK=1,IM                                                     0002025
      Y = ONE(I)*PINV(I,IK)*P(IK,J)                                     0002026
      ZC(J) = ZC(J) + Y                                                 0002027
   45 CALL ROUND(ZC(J),Y,EP)                                            0002028
      ZCB(1) = ZCB(1) - BT(IM,1)                                        0002029
      ZCB(2) = ZCB(2) - BT(IM,2)                                        0002030
  100 RETURN                                                            0002031
C
C USED TO INSURE FEASIABILITY BY MAKING THE B COMPONENT NON-NEGATIVE
C
   50 BT(IM,1) = -BT(IM,1)                                              0002040
      BT(IM,2) = -BT(IM,2)                                              0002050
      DO 60 I=1,K                                                       0002060
   60 DBAR(I) = -DBAR(I)                                                0002070
      PINV(IM,IM) = -1.0                                                0002130
      GO TO 30                                                          0002140
      END                                                               0002150
```

```
C
C
C
0001          SUBROUTINE ROUND(A,B,EP)                                  0002160
      C   A=STARTING VALUE, B=ADDED VALUE, EP= LOWEST ROUND-OFF DESIRED 0002170
0002          IF(A.EQ.0.0.OR.B.EQ.0.0) GO TO 200                        0002180
0003          IF(ABS(A/B).GT.EP) GO TO 200                              0002190
0004          A = 0.0                                                   0002200
0005        200 RETURN                                                  0002210
0006          END                                                       0002220
```

```
      C
      C
      C
0001      SUBROUTINE TIMEIT(N,TIME)                                  0002230
      C
      C   N=0 STARTS CLOCK, N=-1 STOPS CLOCK
      C
0002      IT=N+2                                                     0002270
0003      GO TO (20,10),IT                                           0002280
0004   10 CALL TIMON(M)                                              0002290
0005      TIMEM=M                                                    0002300
0006      RETURN                                                     0002310
0007   20 CALL TIMOFF(M)                                             0002320
0008      TIME=M                                                     0002330
0009      TIME=(TIMEM-TIME)*26.0                                     0002340
0010      RETURN                                                     0002350
0011      END                                                        0002360
```

```
C
C
C
      SUBROUTINE PSEUDO                                                 0002370
     ODIMENSION P(53,151),B(52),C(151),BT(53,2),IBAS(53),ONE(53),TEM(2), 0002380
     1PINV(53,53),ZCB(2),ZC(151),WSB(2),WS(151),T(2),X(53),DBAR(53)     0002390
      COMMON/INT/IUNB,IM,M,N,IBAS,ITER                                  0002400
      COMMON/FLOAT/P,B,C,BT,ONE,TEM,PINV,ZCB,ZC,WSB,WS,T,X,DBAR,EP      0002410
C
C M = NUMBER OF CONSTRAINTS, N = NUMBER OF VARIABLES INCLUDING M SLACK
C    VARIABLES
C MAXIMUM M = 50, MAXIMUM N = 150
C
C CHANGE STATEMENTS 1 AND 2 TO DESIRED PROBLEM SIZE
C
    1 M = 20
    2 N = 70
      IM = M + 1
      IN = N + 1
      K1 = N - M + 1
      K2 = K1 + 1
      C(J) = -URN(1)                                                    0002430
      DO 20 J=2,K
      IF(C(J).LE.-.001) GO TO 10
      C(J) = 0.0                                                        0002450
   10 DO 20 I=2,IM
      P(I,J) = URN(1)                                                   0002470
      IF(P(I,J).GT.0.001) GO TO 20                                      0002480
      P(I,J) = 0.0                                                      0002490
   20 CONTINUE                                                          0002500
      DO 30 I=1,M
      B(I) = URN(1)*5.0                                                 0002520
      IF(B(I).GT.0.01) GO TO 30                                         0002530
      B(I) = 0.0                                                        0002540
   30 CONTINUE                                                          0002550
      DO 40 J=K2,IN
      C(J) =0.0
      DO 40 I = 2,IM
   40 P(I,J) = 0.0                                                      0002580
      DO 50 I=2,IM
      K = N- M + I
   50 P(I,K) = -1.0
      RETURN                                                            0002640
      END                                                               0002650
```

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13. ABSTRACT

Linear programming techniques are becoming of greater importance because the use of computerization has increased the fields for applications for linear programs. The primal-dual algorithm, in which the constraints are added one at a time, is investigated as a possible faster solution method. A computer program was developed to compare this method with the standard primal-dual algorithm using the full set of constraints at one time. Several random problems were solved using these two methods, and the results indicated a significant improvement in the solution time by the use of adding the constraints one at a time.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
LINEAR PROGRAMMING (COMPUTERIZED)						
PRIMAL-DUAL ALGORITHM						